PERGAMON

International Journal of Solids and Structures 39 (2002) 467–481

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

The Lyapunov exponents as a quantitative criterion for the dynamic buckling of composite plates

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Received 1 September 2000; in revised form 1 April 2001

Abstract

The dynamic stability of infinitely wide composite plates subjected to suddenly applied thermal or mechanical loading is investigated. The stability analysis is performed by evaluating the largest Lyapunov exponent, the sign of which characterizes the nature of the response. It is shown that this approach forms an efficient tool which provides a quantitative and unequivocal answer to the question of dynamic buckling of composite plates subjected to various types of loading. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Dynamic stability; Lyapunov exponents; Thermal loading; Mechanical loading; Composite materials

1. Introduction

The stability of motion, as generally defined by Lyapunov, can be considered to be the ability to preserve certain properties of the motion under a perturbation of a specific type. The loss of stability in structures can occur in several ways due to various kinds of dynamic loading. Consequently, the application of the general concept of stability to various problems of dynamic stability of structures yielded numerous approaches that provide several criteria for dynamic buckling. Classification of the numerous problems of dynamic stability into groups and discussion of the applicability of various dynamic stability concepts to different types of structures and loadings can be found in the monograph by Simitses (1990).

Dynamic instability which is caused by periodic loading (termed parametric instability) has been comprehensively studied using a variety of methods. The approach of Lyapunov exponents was suggested for the investigation of this group of problems by Aboudi et al. (1990) where viscoelastic homogeneous plates were considered. It was further employed to study the parametric stability of plates made of several types of homogeneous and composite materials (Cederbaum et al., 1991; Touati and Cederbaum, 1994; Gilat and Aboudi, 2000). This approach is based on the evaluation of a set of numbers, the Lyapunov exponents, the signs of which characterize the nature of the dynamical system. The calculation of these exponents can be efficiently carried out through the incremental procedure suggested by Goldhirsch et al. (1987).

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PII: S0020-7683(01)00108-1

The dynamic stability of structures which are subjected to non-periodic time-dependent loadings have been investigated by many authors, see the monographs by Lindberg and Florence (1987), Jones (1989) and Simitses (1990) for example. For problems of this type, the concept of Budiansky (1967) has been commonly used to determine the stability limit. According to this heuristic concept, the critical conditions under which dynamic buckling occurs involve a large change in the response due to a small change in the loading. Thus, a buckling curve which displays the variation of a characteristic response parameter versus the load parameter, has to be examined. The critical load is that for which the slope of the buckling curve abruptly changes. Nevertheless, the practical application of this dynamic buckling criteria shows that it is not always decisive.

In the present work, the Lyapunov exponents approach is offered as an efficient tool by which the dynamic buckling analysis of composite plates, subjected to a sudden thermal or mechanical loading, can be performed. This approach is based on the determination of the sign of the largest Lyapunov exponent, and thus provides a direct and unequivocal answer to the question whether the response of the plate due to a specific loading is stable or not.

Results for the dynamic buckling of infinitely wide composite plates, obtained using the proposed approach, have been presented. It has been shown that the present analysis is applicable to problems in which the in-plane inertia terms are either included or excluded. The results established by the Lyapunov exponents stability analysis have been compared with those obtained by the application of other concepts of dynamic buckling. Good agreements have been shown demonstrating the advantages of the present approach.

2. Basic formulation

Consider a linearly elastic orthotropic rectangular plate of an infinite width in the y direction, and uniformly supported along the edges $x = 0, L$. The thickness of the plate is h and the coordinate z is perpendicular to the plane of the plate with its origin placed in the mid-plane. The plate is subjected to a suddenly applied loading which is either thermal (namely rapid heating), or mechanical (namely rapid normal in-plane edge displacement or load).

In the framework of the classical plate theory, the von-Karman strains for the present cylindrical bending problem are

$$\varepsilon_{xx} = \varepsilon^0_{xx} + z\varepsilon^1_{xx}, \qquad \varepsilon_{xy} = \varepsilon^0_{xy} \tag{1}$$

where

$$\varepsilon^0_{xx} = u_{x,x} + \tfrac{1}{2}(u_{z,x})^2 + u_{z,x}\, u_{z0,x}, \qquad \varepsilon^0_{xy} = \tfrac{1}{2}u_{y,x}, \qquad \varepsilon^1_{xx} = -u_{z,xx}$$

and u_x, u_y, u_z denote the displacements of a point on the mid-plane, and u_{z0} is the initial geometrical imperfection which is associated with the initial stress-free state. Comma is used for denoting spatial derivatives.

The plate is made of unidirectionally fiber-reinforced layers. Both fibers and matrix are thermo-elastic and anisotropic materials. The effective mechanical and thermal properties of the unidirectional composite are determined by the micromechanical method of cells (Aboudi, 1991). In the framework of this method the overall composite constitutive law is obtained in terms of the material properties of the constituents, and expressed with respect to material coordinates, one of which coincides with the fiber direction. A further transformation reduces this constitutive law to the the plate coordinate system (x, y, z), such that it can be presented in the following form:

$$\boldsymbol{\sigma} = \boldsymbol{C}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{T}}) \tag{2}$$

Here $\boldsymbol{C}$ is the effective elastic stiffness tensor of the composite (which is solely determined by the properties of its constituents), $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the average stress and strain tensors, respectively. The thermal strain of the composite is $\boldsymbol{\varepsilon}^{\mathrm{T}} = \boldsymbol{\alpha}\Delta T$ where $\boldsymbol{\alpha}$ is the effective thermal expansion tensor, and ΔT is the temperature deviation from a reference temperature T_{R} at which the material is strain-free.

2.1. In-plane inertia neglected

Under circumstances for which the in-plane inertia effects are negligible (e.g. distributed surface heating or relatively slow mechanical loading) the equations which govern the motion of the plate are (Reddy, 1997)

$$N_{xx,x} = 0, \qquad N_{xy,x} = 0, \qquad M_{xx,xx} + N_{xx}(u_{z,xx} + u_{0z,xx}) = I_1 \ddot{u}_z \tag{3}$$

where dot and double dot denote first and second time derivatives respectively. The force, moment and inertia resultants, $\boldsymbol{N}$, $\boldsymbol{M}$, and I_1 are given by

$$\begin{aligned} (\boldsymbol{N}, \boldsymbol{M}) &= \int_{-h/2}^{h/2} \boldsymbol{\sigma}(1, z)\mathrm{d}z \\ I_1 &= \int_{-h/2}^{h/2} \rho\, \mathrm{d}z \end{aligned} \tag{4}$$

with ρ being the effective mass density of the composite. For a plane-stress state, these definitions yield the following plate constitutive relations

$$\begin{bmatrix} N_{xx} \\ N_{xy} \\ M_{xx} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{16} & B_{11} \\ A_{16} & A_{66} & B_{16} \\ B_{11} & B_{16} & D_{11} \end{bmatrix} \begin{bmatrix} \varepsilon^0_{xx} \\ 2\varepsilon^0_{xy} \\ \varepsilon^1_{xx} \end{bmatrix} - \begin{bmatrix} N^{\mathrm{T}}_{xx} \\ N^{\mathrm{T}}_{xy} \\ M^{\mathrm{T}}_{xx} \end{bmatrix} \tag{5}$$

where the extension, coupling and bending stiffness matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{D}$ are defined as usual by

$$(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{D}) = \int_{-h/2}^{h/2} \boldsymbol{Q}\left(1,\ z,\ z^2\right)\mathrm{d}z \tag{6}$$

with $\boldsymbol{Q}$ being the reduced effective stiffness coefficients.

The thermal resultants are given by

$$\begin{aligned} \left(N^{\mathrm{T}}_{xx}, M^{\mathrm{T}}_{xx}\right) &= \int_{-h/2}^{h/2} \left[Q_{11}\alpha_{xx} + Q_{12}\alpha_{yy} + Q_{16}2\alpha_{xy}\right]\Delta T(1, z)\mathrm{d}z \\ \left(N^{\mathrm{T}}_{xy}\right) &= \int_{-h/2}^{h/2} \left[Q_{16}\alpha_{xx} + Q_{26}\alpha_{yy} + Q_{66}2\alpha_{xy}\right]\Delta T\mathrm{d}z \end{aligned} \tag{7}$$

The first two of Eq. (3) can be solved to find the in-plane displacements of the mid-plane u_x, u_y as follows. Integration of the first two of Eq. (3) in conjunction with the stress–strain relations (2), the stress resultants defined in Eq. (4) for a state of plane stress, and the strain-displacement relation (1), results in

$$\begin{aligned} N_{xx} &= A_{11}\ \varepsilon^0_{xx} + A_{16}\ 2\varepsilon^0_{xy} - B_{11}\ u_{z,xx} - N^{\mathrm{T}}_{xx} = c_0 \\ N_{xy} &= A_{16}\ \varepsilon^0_{xx} + A_{66}\ 2\varepsilon^0_{xy} - B_{16}\ u_{z,xx} - N^{\mathrm{T}}_{xy} = c_1 \end{aligned} \tag{8}$$

where $c_0(t)$, $c_1(t)$ are integration functions with t denoting time.

Let the temperature field be symmetric with respect to $x = L/2$. Furthermore, it is assumed that the variation of the stiffnesses with respect to x, due to the temperature dependence of the material properties, is negligible. The solution of Eq. (8) for ε^0_{xx} and ε^0_{xy} leads to

$$
\begin{aligned}
\varepsilon^0_{xx} &= c_2 + c_3\ u_{z,xx} \\
2\varepsilon^0_{xy} &= c_4 + c_5\ u_{z,xx}
\end{aligned}
\tag{9}
$$

where

$$
\begin{aligned}
c_2(x,t) &= \frac{a_2 - A_{66}N^{\mathrm{T}}_{xx} + A_{16}N^{\mathrm{T}}_{xy}}{a_1} \\
c_3(t) &= \frac{a_3}{a_1} \\
c_4(x,t) &= \frac{a_4 + A_{16}N^{\mathrm{T}}_{xx} - A_{11}N^{\mathrm{T}}_{xy}}{a_1} \\
c_5(t) &= \frac{a_5}{a_1}
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
a_1 &= A^2_{16} - A_{11}A_{66} \\
a_2 &= -A_{66}c_0 + A_{16}c_1 \\
a_3 &= A_{16}B_{16} - A_{66}B_{11} \\
a_4 &= A_{16}c_0 - A_{11}c_1 \\
a_5 &= A_{16}B_{11} - A_{11}B_{16}
\end{aligned}
$$

Eq. (9) in conjunction with strain–displacement relations (1) yield the in-plane displacements of the mid-plane

$$
\begin{aligned}
u_x &= \int c_2\ \mathrm{d}x + c_3\ u_{z,x} - \frac{1}{2}\int u_{z,x}\ (u_{z,x} + 2u_{0z,x})\ \mathrm{d}x + c_7 \\
u_y &= \int c_4\ \mathrm{d}x + c_5\ u_{z,x} + c_6
\end{aligned}
\tag{11}
$$

with c_6 and c_7 representing rigid body motion which can be neglected.

The expressions for c_0 and c_1 are obtained by imposing the in-plane boundary conditions. Two sets of in-plane boundary conditions are considered in the framework of the degenerated formulation.

(a) For a thermal loading, the in-plane displacements of the edges are prevented, i.e.

$$
u_x = u_y = 0 \quad \text{at } x = 0, L \tag{12}
$$

These boundary conditions in conjunction with Eq. (11) yield

$$
\begin{aligned}
c_0 &= \frac{A_{16}c_5 + A_{11}c_3}{L}[u_{z,x}(0) - u_{z,x}(L)] + \frac{A_{11}}{2L}\int_0^L u_{z,x}(u_{z,x} + 2u_{0z,x})\mathrm{d}x - \frac{1}{L}\int_0^L N^{\mathrm{T}}_{xx}\mathrm{d}x \\
c_1 &= \frac{A_{66}c_5 + A_{16}c_3}{L}[u_{z,x}(0) - u_{z,x}(L)] + \frac{A_{16}}{2L}\int_0^L u_{z,x}(u_{z,x} + 2u_{0z,x})\mathrm{d}x - \frac{1}{L}\int_0^L N^{\mathrm{T}}_{xy}\mathrm{d}x
\end{aligned}
\tag{13}
$$

(b) For a mechanical step loading, it is assumed that

$$
u_x = U(t), \qquad N_{xy} = 0 \quad \text{at } x = 0, L \tag{14}
$$

where $U(t)$ specifies the temporal dependence of the applied displacement. This results in the following expression for c_0 and c_1

$$c_0 = -\frac{a_1 c_3}{A_{66}L}[u_{z,x}(0) - u_{z,x}(L)] - \frac{a_1}{2A_{66}L}\int_0^L u_{z,x}(u_{z,x} + 2u_{0z,x})\mathrm{d}x - \frac{1}{L}\int_0^L N_{xx}^{\mathrm{T}}\mathrm{d}x + \frac{A_{16}}{A_{66}L}\int_0^L N_{xy}^{\mathrm{T}}\mathrm{d}x + \frac{2a_1}{A_{66}L}U(t)$$
$$c_1 = 0 \tag{15}$$

Thus, in both cases the force and moment resultants can be expressed in terms of the plate stiffnesses, thermal stress resultants and the unknown transverse displacement, u_z, such that the third of the equations of motion (3) becomes a partial differential equation that governs the transverse displacement only:

$$[B_{11}(c_2 + c_3 u_{z,xx}) + B_{16}(c_4 + c_5 u_{z,xx}) - D_{11}u_{z,xx} - M_{xx}^{\mathrm{T}}]_{,xx} + c_0(u_{z,xx} + u_{0z,xx}) = I_1\ddot{u}_z \tag{16}$$

Furthermore, in both cases the edges are taken to be simply supported, namely having the following out-of-plane boundary conditions

$$u_z = M_{xx} = 0, \quad \text{at } x = 0, L \tag{17}$$

Let the initial imperfection and the transverse displacement be expressed by the following series

$$u_{0z} = \sum_{j=j_{\min}}^{j_{\max}} W_{0j}\sin(j\pi x/L)$$
$$u_z = \sum_{j=j_{\min}}^{j_{\max}} W_j(t)\sin(j\pi x/L) + \tilde{W}(t)[\cos(2\pi x/L) - 1] \tag{18}$$

where $j_{\min}, j_{\max}$ denote the first and last terms of the series, respectively. These expressions satisfy the first of boundary conditions (17), and $\tilde{W}(t) = B_{11}c_2(0,t) + B_{16}c_4(0,t) - M_{xx}^{\mathrm{T}}$ is defined such that the boundary condition on M_{xx} is fulfilled as well. When symmetry (with respect to z) of the layup and the temperature field exists, $\tilde{W}(t) = 0$.

Application of the Galerkin method with respect to the spatial coordinate x, with $\sin(k\pi x/L)$ as weighting functions, reduces Eq. (16) to the following set of nonlinear ordinary differential equations

$$\ddot{W}_k - \sum_{j=j_{\min}}^{j_{\max}} W_j^2\, W_k\, b_{1kj} - \sum_{j=j_{\min}}^{j_{\max}} W_j^2\, b_{2kj} - \sum_{j=j_{\min}}^{j_{\max}} W_j\, W_k\, b_{3kj} - \sum_{j=j_{\min}}^{j_{\max}} W_j\, b_{4kj} - W_k\, b_{5k} - b_{6k} = 0, \quad k = j_{\min}, \ldots, j_{\max} \tag{19}$$

where b_{1kj}, b_{2kj}, b_{3kj}, b_{4kj}, b_{5k} and b_{6k} are functions of the plate stiffnesses, thermal resultants and the amplitudes of the initial imperfection. These are given in Appendix A.

The differential Eq. (19) are accompanied by the following initial conditions

$$W_j = \dot{W}_j = 0, \quad \text{at } t = 0 \tag{20}$$

2.2. In-plane inertia included

Having applied the Lyapunov exponent approach to problems in which the in-plane inertia terms have been neglected, a formulation which includes these terms is considered next. The effects of the in-plane inertia on the structure response have been discussed by Lindberg and Florence (1987) and McIvor and Bernard (1973) for example. The fully dynamic governing equations of the plate, which in contrast to Eq. (3) include the in-plane inertia terms, are (Reddy, 1997)

$$N_{xx,x} = I_1\ddot{u}_x, \quad N_{xy,x} = I_1\ddot{u}_y, \quad M_{xx,xx} + [N_{xx}(u_{z,x} + u_{0z,x})]_{,x} = I_1\ddot{u}_z \tag{21}$$

By implementing the finite difference method with respect to the spatial coordinate x, using central derivative approximations, the nonlinear partial differential equations (21) are reduced to a set of nonlinear ordinary differential equations with respect to time

$$\ddot{\boldsymbol{U}} = \boldsymbol{L}(\boldsymbol{U}) \tag{22}$$

Here

$$\boldsymbol{U} = [U_{x_1}, U_{x_2}, \ldots, U_{x_{p\max-1}}, U_{y_1}, U_{y_2}, \ldots, U_{y_{p\max-1}}, U_{z_1}, U_{z_2}, \ldots, U_{z_{p\max-1}}]^{\mathrm{T}}$$

with $U_{x_p} = u_x(x_p, t)$, $U_{y_p} = u_y(x_p, t)$, $U_{z_p} = u_z(x_p, t)$ being the values of the dependent variables at the spatial mesh points x_p, $p = 0, 1, \ldots, p_{\max}$ (thus $p_{\max} - 1$ is the number of spatial intervals which is determined such that convergence is achieved), and $\boldsymbol{L}(\boldsymbol{U})$ are nonlinear functions of $\boldsymbol{U}$.

These equations are accompanied by the out-of-plane boundary conditions (17), and by the following in-plane boundary conditions

$$N_{xx} = N(t), \qquad N_{xy} = 0 \quad \text{at } x = 0, L \tag{23}$$

through which U_{x_0}, $U_{x_{p\max}}$, U_{y_0}, $U_{y_{p\max}}$, U_{z_0}, $U_{z_{p\max}}$ are defined. Here $N(t)$ specifies the temporal dependence of the applied axial load. The initial conditions are

$$\boldsymbol{U} = \dot{\boldsymbol{U}} = 0, \quad \text{at } t = 0 \tag{24}$$

3. Dynamic stability analysis

In order to investigate the stability of composite plates under a suddenly applied thermal or mechanical load the concept of Lyapunov exponents is employed. Lyapunov stability analysis of a dynamical system consists of the evaluation of a corresponding set of characteristic numbers (e.g. Hahn, 1967). The negative values of these characteristic numbers are known as Lyapunov exponents. According to Lyapunov, the motion is asymptotically stable if all the exponents are negative. A positive Lyapunov exponent indicates an exponential separation between two initially close trajectories, namely instability of the system (Chetaev, 1961). The system is stable if the largest Lyapunov exponent is not greater than zero. Consequently, it is sufficient to evaluate the largest Lyapunov exponent in order to characterize the behavior of a dynamical system.

According to Goldhirsch et al. (1987) the Lyapunov exponents can be efficiently determined by the following procedure. Consider the system of ordinary nonlinear differential equations

$$\dot{\boldsymbol{v}} = \mathbf{F}(\boldsymbol{v}) \tag{25}$$

The associated stability equation is given by

$$\dot{\boldsymbol{y}} = \boldsymbol{G}\boldsymbol{y}, \quad G_{km} = \left.\frac{\partial F_k}{\partial v_m}\right|_{\boldsymbol{v}=\boldsymbol{v}(t)} \tag{26}$$

and $\boldsymbol{y}$ can be regarded as a small perturbation of $\boldsymbol{v}$. Within the time increment $0 < t < t^{(1)}$, the system (26) with $\boldsymbol{G}(t = 0)$ is solved numerically for the normalized initial conditions

$$||\boldsymbol{y}(0)|| = 1$$

where $||.||$ is the Euclidean norm. This yields $\boldsymbol{y}(t^{(1)})$.

Eq. (26) with $\boldsymbol{G}(t = t^{(1)})$ and with the following initial conditions

$$\boldsymbol{y}(t^{(1)}) = \boldsymbol{z}(t^{(1)}), \quad \boldsymbol{z}(t^{(1)}) = \frac{\boldsymbol{y}(t^{(1)})}{||\boldsymbol{y}(t^{(1)})||}$$

are then solved within the second time interval $t^{(1)} < t < t^{(2)}$ yielding $\boldsymbol{y}(t^{(2)})$. The process is repeated for n time intervals while correspondingly, the system (25) is solved to provide the values of $\boldsymbol{v}(t^{(n)})$ needed for the evaluation of $\boldsymbol{G}$. Namely, the incremental procedure is simultaneously used to get the nonlinear response and the Lyapunov exponents.

For the nth time interval, let us define the value of the parameter μ_n as follows:

$$\mu_n = \sum_{l=1}^{n} \log_e ||\boldsymbol{y}(t^{(l)})||/t^{(n)} \tag{27}$$

It has been shown (Goldhirsch et al., 1987) that for finite large time, the value of μ_n approaches to the value of the Lyapunov exponent.

Employing the above procedure to investigate the dynamic stability of elastic infinitely wide plate, Eq. (25) is obtained by reducing Eq. (19) or Eq. (22) to a set of first order differential equations. The matrix $\boldsymbol{G}$ in the stability equations (26) can thus be written as follows:

$$\boldsymbol{G} = \begin{bmatrix} [\boldsymbol{0}] & [\boldsymbol{I}] \\ [\boldsymbol{g}] & [\boldsymbol{0}] \end{bmatrix} \tag{28}$$

where the square sub-matrices $\boldsymbol{0}$, $\boldsymbol{I}$ are the zero and unit matrices respectively. The order of these sub-matrices and the specific form of sub-matrix $\boldsymbol{g}$ depends on the adopted formulation as described below.

When in-plane inertia effects are excluded and the governing equations are given by Eq. (19), each of the sub-matrices of $\boldsymbol{G}$ is of order $\overline{J} = j_{\max} - j_{\min} + 1$, where $\overline{J}$ is the number of terms in the series approximation Eq. (18), and the elements of $\boldsymbol{g}$ have the following form:

$$g_{km} = \delta_{km}\left[\sum_{j=j_{\min}}^{j_{\max}} W_j^2 b_{1kj} + \sum_{j=j_{\min}}^{j_{\max}} W_j b_{3kj} + b_{5k}\right] + 2W_k W_m b_{1km} + 2W_m b_{2km} + W_k b_{3km} + b_{4km} \tag{29}$$

with δ_{km} being the Kronecker delta.

When in-plane inertia effects are included and Eq. (22) is relevant, each of the sub-matrices of $\boldsymbol{G}$ is of order $3(p_{\max} - 1)$.

Note that the matrix $\boldsymbol{g}$ is a function of the stiffnesses which may be temperature dependent, and of the displacement variables. The latter are the solution of the ordinary nonlinear differential equations (19) or (22), which have been reduced to a set of first order equations in order to confirm with the general form (25). Hence Eq. (19) or Eq. (22), have to be solved simultaneously with the progressing of the stability analysis, such that at the end of each time increment, the elements of the matrix $\boldsymbol{G}$ are updated accordingly, as has been discussed above.

4. Application

The previously described method for evaluating the largest Lyapunov exponent is employed herein in order to determine whether the behavior of an infinitely wide elastic composite plate subjected to a sudden thermal or mechanical loading is stable or not. Thus the question of dynamic stability of the composite plate is quantitatively determined. The established results of the Lyapunov exponents stability analysis are compared with those obtained by the application of the concept of dynamic buckling of Budiansky (1967). The generated results are based on a one term approximation of the displacement, namely with

$j_{\min} = j_{\max} = 1$ in Eq. (18). For the fully dynamic analysis including in-plane inertia, spatial and time increments were chosen such as to assure numerical convergence. The material properties of the matrix and fiber phases are: $E^{\mathrm{m}} = 72.4$ GPa, $\nu^{\mathrm{m}} = 0.33$, $\alpha^{\mathrm{m}} = 23.1 \times 10^{-6}$ per °C for the matrix, and $E^{\mathrm{f}} = 414$ GPa, $\nu^{\mathrm{f}} = 0.3$, $\alpha^{f} = 4.8 \times 10^{-6}$ per °C for the fibers (E, ν and α denote the Young's modulus, Poisson's ratio and coefficient of thermal expansion). The thickness ratio of the plate is $L/h = 100$. In the presented results time is given in seconds.

4.1. Thermal loading

Considered a unidirectional plate with fiber oriented in an angle of 45° with respect to the x axis, having initial imperfection $W_{01} = 0.001$ h and being initially at a temperature $T_{\mathrm{R}} = 21$°C. Let the plate's upper and lower surfaces be subjected to a sudden uniform time-dependent heating of the form: $T(t) = T_{\mathrm{R}} + T_m(1 - \mathrm{e}^{-500t})$ with T_m being the temperature amplitude. For thin plates it can be assumed that the resulting temperature field is spatially uniform. Under these loading conditions the effects of the in-plane inertia are negligible and the formulation which is given in Section 2.1(a) is used.

4.1.1. Linear analysis

Neglecting terms containing high order powers of W_n, Eq. (19) becomes the linear equation of motion of the plate, which for one mode approximation has the following form:

$$\ddot{W}_k - W_k(b_{4kk} + b_{5k}) - b_{6k} = 0 \tag{30}$$

Disregarding the temporal dependence of b_{4kk}, b_{5k} (which is due to the time-dependent applied temperature) reduces Eq. (30) to a linear second order ordinary differential equation with constant coefficients. The nature of the solution of Eq. (30) depends on the sign of the coefficient of W_k (Jones, 1989). The thermal buckling load, separating the bounded motion (trigonometric solutions of Eq. (30)) and the unbounded motion (hyperbolic solutions of Eq. (30)) is that for which

$$b_{4kk} + b_{5k} = 0.$$

This is actually the static thermal buckling load, which in the present case leads to the following expression for the critical temperature amplitude

$$T_{m\,\mathrm{cr}} = \frac{D_{11}\pi^2}{L^2(A_{11}\alpha_{xx} + A_{22}\alpha_{yy} + 2A_{12}\alpha_{xy})}$$

yielding, for the present case, the value: $T_{m\,\mathrm{cr}} = 4.074$°C.

The response of the plate, as predicted by the linear analysis, is shown in Fig. 1 for thermal loads of two different temperature amplitudes: $T_m = 4.0$°C, and $T_m = 4.1$°C. It is readily observed that the response caused by the thermal loading with $T_m = 4.0$°C is bounded whereas the response with $T_m = 4.1$°C is unbounded, implying that the system under the latter loading condition, ($T_m = 4.1$°C) is unstable. The dynamic thermal buckling load, namely the largest amplitude of the thermal loading which yields a bounded response is therefore $T_{m\,\mathrm{cr}} = 4.0$°C.

In Fig. 2, the variation with time of the largest Lyapunov exponent of the plate under the aforementioned loading conditions is presented. Under a thermal loading of amplitude $T_m = 4.0$°C, the largest Lyapunov exponent approaches zero (Fig. 2a), indicating a stable response. For $T_m = 4.1$°C, on the other hand, the Lyapunov exponent is positive (Fig. 2b), implying instability of the system. Hence, the Lyapunov exponent stability analysis of the thermally induced motion of the linearly elastic plate agrees with the conclusions drawn from the direct examination of the plate response and its nature (Fig. 1), under loads of various amplitudes. Consequently, the Lyapunov exponents form a quantitative and effective tool for investigating and determining whether the dynamic response of the plate is stable or not.

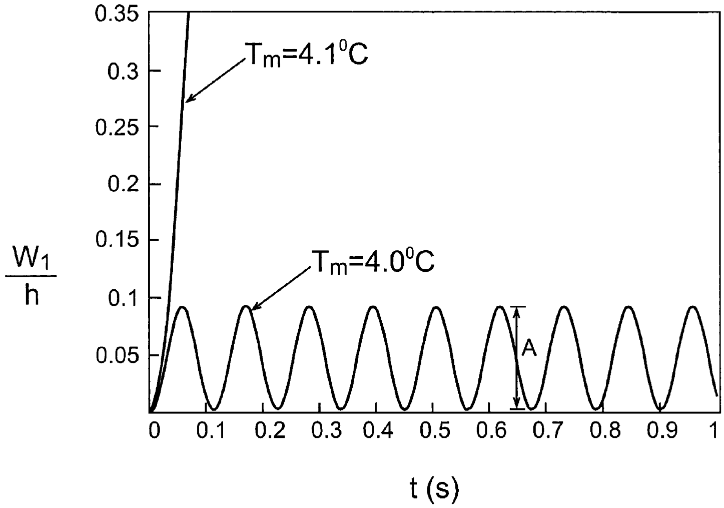

Fig. 1. Variation of the maximum transverse displacement with time for a unidirectional [45°] plate subjected to a thermal load of amplitude T_m (linear analysis).

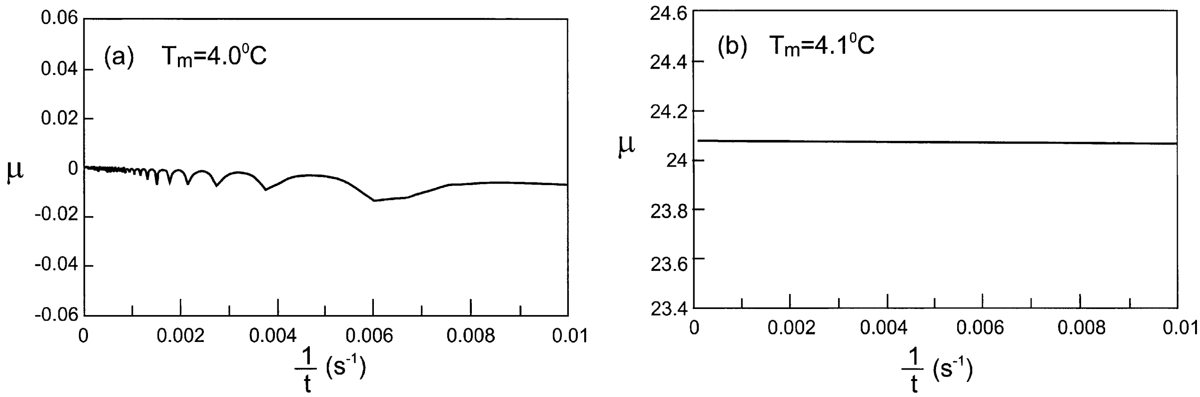

Fig. 2. Lyapunov exponents for a unidirectional [45°] plate subjected to a thermal load of amplitude T_m (linear analysis).

4.1.2. Geometrically nonlinear analysis

Having studied the linearized problem of dynamic buckling, let us investigate the nonlinear equations of motion (19) that incorporate the geometrically nonlinear effects. In Fig. 3, the response of the plate, as predicted by the nonlinear analysis, is shown for thermal loads with two different temperature amplitudes: $T_m = 3.9$°C, and $T_m = 4.0$°C. It is seen that both loadings yield bounded motion, and this is true also for loads of lower or somewhat higher amplitudes (not shown here). However, contrary to the linear case, a bounded response does not necessarily imply stability.

In order to check the stability of the thermally induced nonlinear motion, the largest Lyapunov exponents for the plate under the same type of thermal loads are examined. These are shown in Fig. 4. According to Fig. 4a, the plate subjected to a load of amplitude $T_m = 3.9$°C exhibits a stable behavior, since its Lyapunov exponent approaches zero. Under a load of temperature amplitude $T_m = 4.0$°C, on the other hand, the plate is unstable since its Lyapunov exponent, shown in Fig. 4b, approaches a positive value.

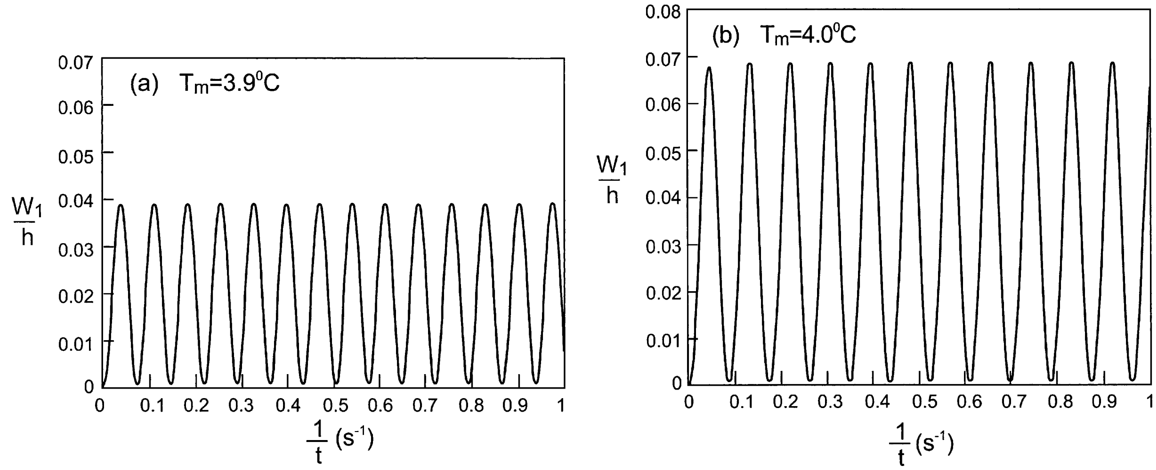

Fig. 3. Variation of the maximum transverse displacement with time for a unidirectional [45°] plate subjected to a thermal load of amplitude T_m (nonlinear analysis).

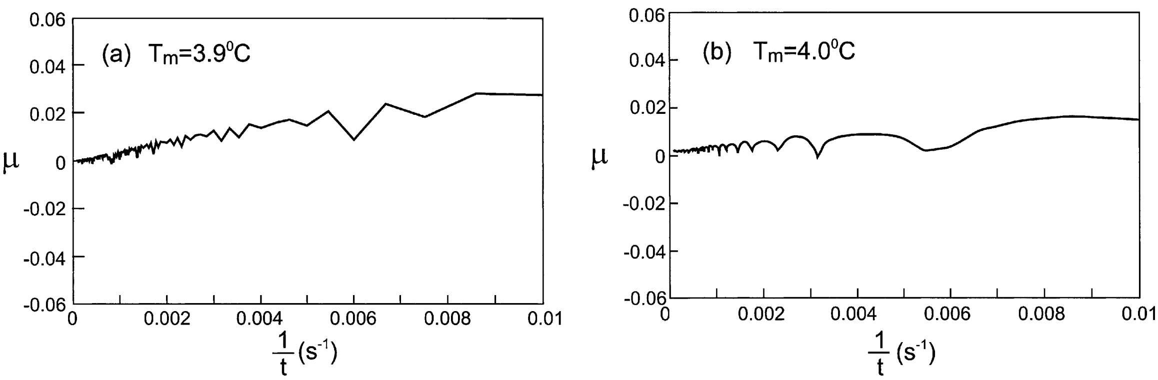

Fig. 4. Lyapunov exponents for a unidirectional [45°] plate subjected to a thermal load of amplitude T_m (nonlinear analysis).

As demonstrated in the previous subsection, the loss of stability of a linear system can be characterized also by the development of a divergent motion, namely an unbounded response. In contrast, it turns out that the nonlinear response of the plate due to thermal loading remains bounded, though, not necessarily stable. The definition of the stability limit in these cases, has to be based on another criterion. One of the concepts used in various investigations (e.g. Gilat and Aboudi, 1995; Ari-Gur and Simonetta, 1997) is the stability concept of Budiansky (1967). According to this concept, the critical conditions under which dynamic buckling occurs involve a large change in the response due to a small change in the loading. In order to employ Budiansky's criterion, a buckling curve which displays the variation of a characteristic response parameter versus the load parameter, has to be investigated. The critical load is then determined by inspection as the one for which the slope of the buckling curve abruptly changes.

Fig. 5 shows the buckling curves for the previously discussed plate as predicted by the linear and the nonlinear analyses. The figure shows the variation of the non-dimensional amplitude A of the thermally induced vibrations with the increase of the thermal loading amplitude T_m. The buckling curve resulting from the linear analysis in conjunction with the Budiansky's concept yields the same thermal buckling load which has been determined before, using the Lyapunov exponent.

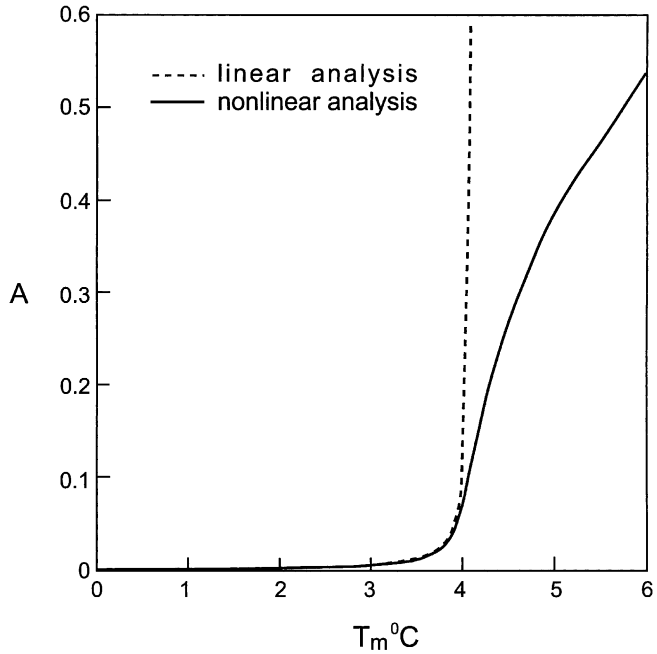

Fig. 5. Buckling curves for a unidirectional [45°] plate subjected to a thermal load of amplitude T_m.

Trying to apply Budiansky's concept for the nonlinear analysis, it turns out that this criterion for dynamic instability is not always decisive. It becomes ambiguous when the change of the slope of the buckling curve is not abrupt. Moreover, it may depend on the choice of the parameter characterizing the structure dynamical response (such as A).

A comparison between the observations resulting from the Lyapunov exponent analysis and those based on Budiansky's concept, however, shows consistency. The Lyapunov exponent analysis on the other hand clearly distinguishes stable systems from unstable ones. In addition, it enables the direct diagnosis of instability of the motion of a plate due to a specific loading, without the awkward need to determine its response over a wide spectrum of loading parameters.

4.2. Mechanical loading

4.2.1. Step load

Basing on the formulation given in Section 2.1(b), consider a three layered symmetric angle-ply $[45°/-45°/45°]$ the edges $x = 0, L$ of which are subjected to a sudden normal in-plane displacement

$$U(t) = U_m U_0(t) \tag{31}$$

where

$$\begin{aligned} &U_0(t) = [t^3 H(t) - 3t_1^3 H(t_1) + 3t_2^3 H(t_2) - t_3^3 H(t_3)]/(6\eta^3) \\ &t_1 = t - \eta, \quad t_2 = t - 2\eta, \quad t_3 = t - 3\eta \end{aligned} \tag{32}$$

where $H(t)$ is the Heaviside unit function. According to this equation the edge displacement rises smoothly from zero at time $t = 0$ to the value U_m at $t = 3\eta$, and remains constant thereafter.

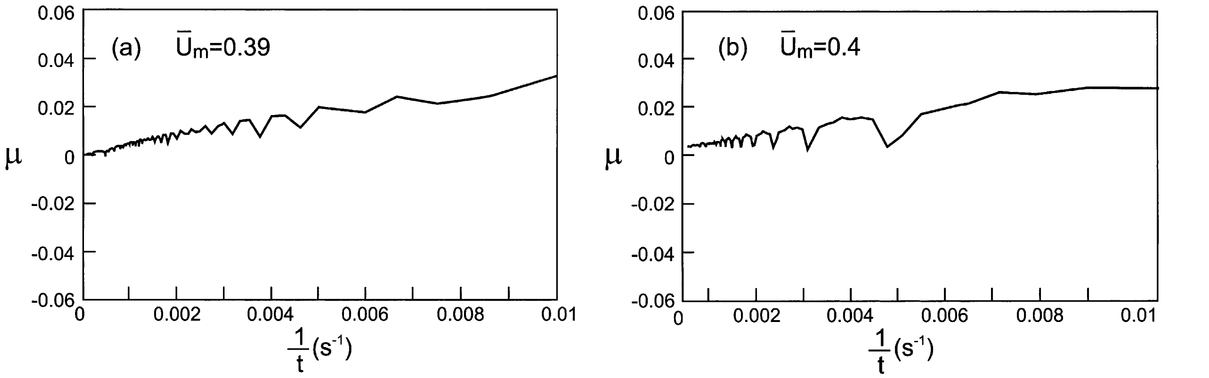

Fig. 6. Lyapunov exponents for a [45°/ − 45°/45°] plate subjected to a mechanical step load through edge shortening of amplitude U_m.

Fig. 6 displays the Lyapunov exponents of the plate subjected to the previously described mechanical loading with $\eta = 0.02$ s as predicted by the nonlinear analysis. It is readily observed that whereas the loading with an edge displacement amplitude $\overline{U}_m = U_m 10^4/L = 0.39$ results in a stable motion (which is characterized by a vanishing Lyapunov exponent (Fig. 6a)), the loading amplitude $\overline{U}_m = 0.4$ yields an unstable response (with is characterized by a positive Lyapunov exponent (Fig. 6b)). It should be noted that both loadings induce bounded motion.

4.2.2. Impulsive load of finite duration

Consider a unidirectional [0°] plate the edges $x = 0, L$ of which are subjected to an axial load of the following form

$$N(t) = -N_m[U_0(t) - U_0(t - 3\eta)]$$

which describes an impulse of amplitude N_m and duration 6η. The presented results are for $\eta = 1.23\ 10^{-4}$ s.

Results of the Lyapunov exponent stability analysis based on the fully dynamic equations (including in-plane inertia) are presented in Fig. 7. An impulsive loading of amplitude $\overline{N}_m = N_m/h = 24.$ MPa yields a

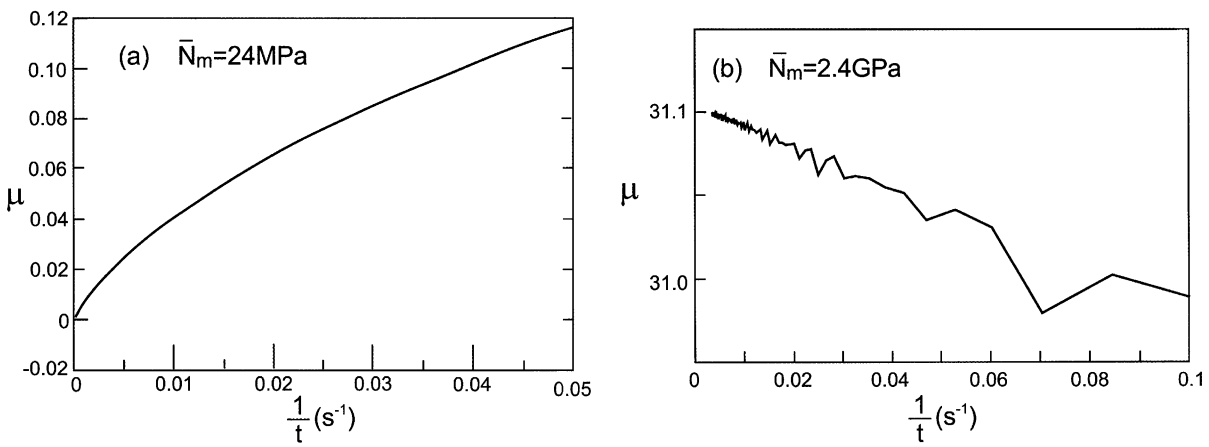

Fig. 7. Lyapunov exponents for a unidirectional [0°] plate subjected to a mechanical impulse of amplitude N_m.

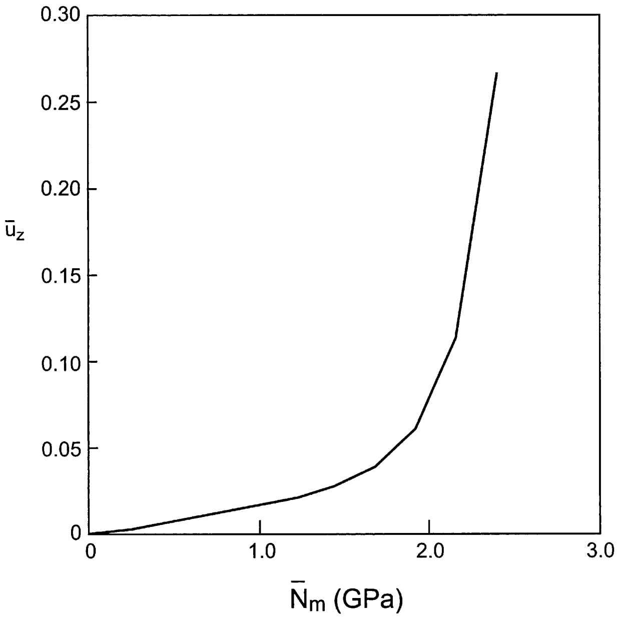

Fig. 8. Buckling curve for a unidirectional [0°] plate subjected to a mechanical impulse of amplitude N_m.

Lyapunov exponent approaching zero (Fig. 7a) namely a stable response. On the other hand, a load of amplitude $\overline{N}_m = 2.4$ GPa yields a positive Lyapunov exponent (Fig. 7b) indicating an unstable motion.

In order to compare the results of the Lyapunov exponent stability analysis with those based on the concept of Budiansky, the buckling curve, as predicted by the fully dynamic analysis, is presented in Fig. 8. This shows the variation of the non-dimensional maximum transverse displacement, $\overline{u}_z = \max_t \max_x u_z/h$, with the load amplitude. The observation drawn from Fig. 7 can be considered to be consistent with the character of the corresponding buckling curve.

5. Conclusions

The dynamic stability of elastic composite plates has been quantitatively determined by means of Lyapunov exponents. The results of the Lyapunov exponent analysis were shown to be consistent with the observations based on other qualitative criteria for dynamic buckling. However, the Lyapunov exponent concept is decisive and provides an efficient tool for clearly distinguishing stable systems from unstable ones. Moreover, it enables the direct diagnosis of the nature of the plate response to a specific loading.

In the preset work the stability of infinitely wide linearly elastic composite plates was investigated in the framework of geometrically nonlinear analysis. The Lyapunov exponents concept should be suitable for studying the stability of structural systems having a different geometry, such as shells, or made of material exhibiting nonlinearly elastic or viscoelastic behavior.

Acknowledgements

The second author gratefully acknowledges the support of the Diana and Arthur Belfer chair of Mechanics and Biomechanics.

Appendix A

Taking into account the symmetry with respect to $x = L/2$ (implying that in Eq. (18) odd terms only are included), the functions in Eq. (19) have the following form:

$$
\begin{aligned}
I_1 b_{1kj} &= -\left(\frac{k\pi}{L}\right)^2 d_{2j} \\
I_1 b_{2kj} &= -\left[W_{0k}\left(\frac{k\pi}{L}\right)^2 + \tilde{W}\frac{16k\pi}{L^2(k^2-4)}\right] d_{2j} - \frac{4k\pi}{L^2}(d_{2j}c_3 + d_{5j}c_5) \\
I_1 b_{3kj} &= -\left(\frac{k\pi}{L}\right)^2 d_{1j} \\
I_1 b_{4kj} &= -\left[W_{0k}\left(\frac{k\pi}{L}\right)^2 + \tilde{W}\frac{16k\pi}{L^2(k^2-4)}\right] d_{1j} - \frac{4k\pi}{L^2}(d_{1j}c_3 + d_{4j}c_5) \\
I_1 b_{5k} &= -\left(\frac{k\pi}{L}\right)^2 d_3 - \left(\frac{k\pi}{L}\right)^4 (B_{11}c_3 + B_{16}c_5 - D_{11}) - \left(\frac{k\pi}{L}\right)^2 \frac{2a_1}{A_{66}L} U(t) \\
I_1 b_{6k} &= -W_{0k}\left(\frac{k\pi}{L}\right)^2 d_3 + \tilde{W}\frac{16k\pi}{L^2(k^2-4)}\left[d_3 - \frac{k\pi}{L}(B_{11}c_3 + B_{16}c_5 - D_{11})\right] - \frac{4k\pi}{L^2}(d_3c_3 + d_6c_5) \\
&\quad - \frac{2k^2\pi^2}{L^3}\left[c_3\int N_{xx}^{\mathrm{T}}\sin\left(\frac{k\pi x}{L}\right)\mathrm{d}x + c_5\int N_{xy}^{\mathrm{T}}\sin\left(\frac{k\pi x}{L}\right)\mathrm{d}x - \int M_{xx}^{\mathrm{T}}\sin\left(\frac{k\pi x}{L}\right)\mathrm{d}x\right] \\
&\quad - \frac{16}{\pi k(k^2-4)}\ddot{\tilde{W}} - \left[W_{0k}\left(\frac{k\pi}{L}\right)^2 + \tilde{W}\frac{16k\pi}{L^2(k^2-4)} + \frac{4k\pi}{L^2}c_3\right]\frac{2a_1}{A_{66}L}U(t)
\end{aligned}
$$

for thermal loading $(U(t) = 0)$

$$
\begin{aligned}
d_{1j} &= \frac{2j\pi}{L^2}(A_{11}c_3 + A_{16}c_5) + \frac{j^2\pi^2}{2L^2}W_{0j}A_{11} - \frac{8j\pi}{(4-j^2)L}\tilde{W}A_{11} \\
d_{2j} &= \left(\frac{j\pi}{2L}\right)^2 A_{11} \\
d_3 &= -\tilde{W}A_{11}\sum_j W_{0j}\frac{8j\pi}{(4-j^2)L} + \left(\frac{\pi}{L}\right)^2\tilde{W}^2A_{11} - \frac{1}{L}\int N_{xx}^{\mathrm{T}}\mathrm{d}x \\
d_{4j} &= \frac{2j\pi}{L^2}(A_{16}c_3 + A_{66}c_5) + \frac{j^2\pi^2}{2L^2}W_{0j}A_{16} - \frac{8j\pi}{(4-j^2)L}\tilde{W}A_{16} \\
d_{5j} &= \left(\frac{j\pi}{2L}\right)^2 A_{16} \\
d_6 &= -\tilde{W}A_{16}\sum_j W_{0j}\frac{8j\pi}{(4-j^2)L} + \left(\frac{\pi}{L}\right)^2\tilde{W}^2A_{16} - \frac{1}{L}\int N_{xy}^{\mathrm{T}}\mathrm{d}x
\end{aligned}
$$

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